Bianchi - Euler system for relativistic fluids and Bel - Robinson type energy.

Yvonne Choquet-Bruhat and James W. York

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Abstract

We write a first order symmetric hyperbolic system coupling the Riemann tensor with the dynamical acceleration of a prefect relativistic fluid. We determine the associated, coupled, Bel - Robinson type energy, and the integral equality that it satisfies.

Système de Bianchi-Euler pour un fluide relativiste, et énergie de type Bel-Robinson.

Résumé. On écrit un système symétrique hyperbolique satisfait par le tenseur de Riemann de l'espace temps et l'accélération dynamique d'un fluide parfait relativiste. On détermine l'énergie du type Bel-Robinson correspondante, et l'égalité intégrale qu'elle satisfait.

1 Introduction.

The effective strength of the gravitational field lies in the Riemann tensor of the spacetime metric. Its evolution is governed by the so - called higher order equations (Bel 1958, Lichnerowicz 1964), deduced from the Bianchi identities. The system satisfied by the trace free part of the Riemann tensor, the Weyl tensor, was some time ago recognized as a linear, first order symmetric hyperbolic system (FOSH), with constraints, homogeneous in vacuum. See H. Friedrich (1996) and references therein. The evolution equations for the Riemann tensor itself have also been written (CB-Yo 1997) as a FOSH system, made explicit in terms of four two- tensors, introduced by Bel (1958)

in a general setting, the electric and magnetic gravitational fields and corresponding inductions relative to a Cauchy adapted frame. In the presence of sources the higher order equations, now renamed *Bianchi equations*, are no longer homogeneous; their right hand sides are linear in the covariant derivative of the stress energy tensor of the sources.

In this article we consider the case of perfect fluid sources. The Einstein equations with such sources have long ago (CB 1958) been proved to be a well posed Leray - hyperbolic system (with constraints). The fluid equations have also been written as a FOSH system (in special relativity, K.O. Friedrichs 1969; in general relativity Ruggeri and Strumia 1981, Anile 1982, and Rendall 1992 who proved a theorem valid also for isolated fluid bodies with special equations of state). It seemed however interesting to have a system of equations which would be a FOSH system both for the gravitational field, namely the Riemann tensor of space time, and the fluid variables. Such a system has been written in lagrangian variables (that is, in a frame whose timelike axis is tangent to the fluid flow lines) by H. Friedrich 1998, who used the Weyl tensor and by CB-Yo 2001, using directly the Riemann tensor. In this paper we use eulerian variables, that is a frame adapted to the usual 3+1 slicing of the space time, with time axis orthogonal to the space slices, which we call a Cauchy adapted frame. We obtain a FOSH system for the Riemann tensor and the dynamical fluid acceleration. The energy corresponding to the FOSH system that we obtain, and call the Bianchi - Euler system, is the sum of the usual Bel Robinson energy of the gravitational field, and the dynamical acceleration energy of the fluid. It is not conserved in general - no more than the Bel - Robinson energy in vacuum - but its evolution can be controlled through the equations we obtain.

2 Einstein equations with fluid sources.

The Einstein equations with source a stress energy tensor are:

$$R_{\alpha\beta} = \rho_{\alpha\beta}, \quad \rho_{\alpha\beta} \equiv T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T$$

The stress energy tensor of a perfect relativistic fluid is

$$T_{\alpha\beta} \equiv (\mu + p) u_\alpha u_\beta + p g_{\alpha\beta}$$

with u^α the unit kinematical velocity satisfying $u^\alpha u_\alpha = -1$; and μ, p, S the specific energy, pressure and entropy. These thermodynamic quantities are

assumed to be positive. They are linked by an equation of state:

$$\mu = \mu(p, S) \tag{2.1}$$

The enthalpy index f of the fluid is given by the identity

$$f(p, S) \equiv \chi \exp \int_{p_0}^{p} \frac{dp}{\mu(p, S) + p}, \tag{2.2}$$

where χ is a constant such that f^2 has the dimensions of an energy density[1].

The dynamical velocity C^α, a vector tangent to the flow lines, incorporates information on the kinematic velocity u^α and the thermodynamic quantities. It is defined by

$$C^\alpha \equiv f u^\alpha, \tag{2.3}$$

hence it satisfies the relation

$$C^\alpha C_\alpha = -f^2. \tag{2.4}$$

In terms of the dynamical velocity the equations of a perfect fluid in an arbitrary frame are

$$C^\alpha \nabla_\alpha C_\beta + f \partial_\alpha f \equiv C^\alpha (\nabla_\alpha C_\beta - \nabla_\beta C_\alpha) = 0 \tag{2.5}$$

and, with $\mu'_p(p, S) \equiv \partial \mu(p, S)/\partial p$, a given function of p and S,

$$\nabla_\alpha C^\alpha + (\mu'_p - 1) \frac{C^\alpha C^\beta}{C^\lambda C_\lambda} \nabla_\alpha C_\beta = 0 \tag{2.6}$$

$$C^\alpha \nabla_\alpha S = 0. \tag{2.7}$$

In these equations the unknowns are the four components of the vector C^α, and the scalar S. The specific pressure p is a known function of f (i.e. of C^α, by 2.4) and of S

$$p \equiv p(C^\alpha C_\alpha, S) \tag{2.8}$$

[1]That is, in general relativity where time and length have the same dimension and mass-energy the dimension of a length, f and therefore χ have dimension of the inverse of a length.

determined by inverting the relation 2.2.

There seems to be more equations than unknowns, but the equations 2.5. are not independent, because they satisfy the following identity:

$$C^{\alpha}C^{\beta}(\nabla_{\alpha}C_{\beta} - \nabla_{\beta}C_{\alpha}) \equiv 0.$$

The equation 2.7 says that S is constant along the flow lines, hence constant in spacetime if constant initially. We suppose, to simplify what follows below, that S is constant. Removing this hypothesis introduces no essential difficulty, but one has to add as new unknowns the derivatives $\partial_i S$, and the equations obtained from 2.7 by taking a derivative ∇_i. Our results are valid as they stand for a barotropic fluid, because then $\mu \equiv \mu(p)$.

3 Bianchi equations.

We now work in a Cauchy adapted frame, that is, a frame with its timelike axis orthogonal to the space slices. The spacetime metric takes then the usual 3+1 form

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt).$$

The derivatives ∂_α are the Pfaff derivatives in the coframe $\theta^0 = dt, \quad \theta^i = dx^i + \beta^i dt$, that is

$$\partial_0 = \frac{\partial}{\partial t} - \beta^i \frac{\partial}{\partial x^i}, \quad \partial_i = \frac{\partial}{\partial x^i}.$$

We have written in CB.Yo 1997 the Bianchi equations satisfied by the Riemann tensor as a FOSH system

$$\nabla_0 R_{hk,0j} + \nabla_k R_{0h,0j} - \nabla_h R_{0k,0j} = 0 \tag{3.1}$$

and

$$\nabla_0 R^{0}_{:::i,0j} + \nabla_h R^{h}_{:::i,0j} = J_{i,0j} \equiv \nabla_0 \rho_{ji} - \nabla_j \rho_{0i} \tag{3.2}$$

The equations 3.1. and 3.2. are for each given pair $(0j)$ a first order symmetric system, hyperbolic relative to the space sections for the components $R_{hk,0j}$ and $R_{0h,0j}$ because the matrix M^0 of the coefficients of the derivatives

∂_0 is the unit matrix, and the matrix M^t of the coefficients of the derivatives $\frac{\partial}{\partial t}$ is indentical to M^0.

Analogous results hold for the components $R_{hk,ij}$ and $R_{0h,ij}$.

To the Bianchi system is associated its Bel - Robinson energy density on a space slice, namely

$$\mathcal{B} \equiv \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{H}|^2 + |\mathbf{D}|^2 + |\mathbf{B}|^2)$$

where $\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}$ are the electric and magnetic fields and inductions space 2- tensors associated with the Riemann tensor[2]:

$$E_{ij} \equiv R^0{}_{i,0j}, \quad D_{ij} \equiv \frac{1}{4}\eta_{ihk}\eta_{jlm}R^{hk,lm}, \ .$$

$$H_{ij} \equiv \frac{1}{2}N^{-1}\eta_{ihk}R^{hk}{}_{,oj}\,, \quad B_{ji} \equiv \frac{1}{2}N^{-1}\eta_{ihk}R_{0j,}{}^{hk}\,.$$

This energy satisfies the equality

$$\frac{1}{2}\partial_0(|\mathbf{E}|^2 + |\mathbf{H}|^2 + |\mathbf{D}|^2 + |\mathbf{B}|^2) + \bar{\nabla}_h\{N\eta^{lh}{}_i(E^{ij}H_{lj} - B^{ij}D_{lj}\} = Q(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}) + \mathcal{S}$$

where Q is a quadratic form in $\mathbf{E}, ...\mathbf{B}$ with coefficients $\bar{\nabla}N$ and $N\mathbf{K}$, $\mathbf{K}$ extrinsic curvature of the spaceslices. The source term $\mathcal{S}$, zero in vacuum, is

$$\mathcal{S} \equiv J_{0ij}E^{ij} - \frac{1}{2}NJ_{lmi}\eta_h{}^{lm}B^{ih}. \tag{3.3}$$

The tensor $\rho_{\alpha\beta}$ for a perfect fluid is given in terms of C_α by

$$\rho_{\alpha\beta} \equiv (\mu + p)f^{-2}C_\alpha C_\beta + \frac{1}{2}g_{\alpha\beta}(\mu - p) \tag{3.4}$$

The sources of the Bianchi equations are therefore of the first order in C_α, linear in the derivatives $\nabla_\alpha C_\beta$.

[2] η_{ijk} the volume form, $||$and $\bar{\nabla}$, the norm and covariant derivative, are defined by the space metric g_{ij}..

4 Equations for ∇C.

The dynamical acceleration ∇C satisfies the following equations obtained by covariant differentiation of 2.5. and 2.6, and use of the Ricci identities:

$$M_{\gamma\beta} \equiv C^{\alpha}(\nabla_{\alpha} C_{\gamma\beta} - \nabla_{\beta} C_{\gamma\alpha}) + a_{\gamma\beta} = 0 \tag{4.1}$$

and

$$g^{\alpha\beta}\nabla_{\alpha} C_{\gamma\beta} + (\mu'_p - 1)\frac{C^{\alpha}C^{\beta}}{C^{\lambda}C_{\lambda}}\nabla_{\alpha} C_{\gamma\beta} + b_{\gamma} = 0 \tag{4.2}$$

where we have set

$$C_{\gamma\beta} \equiv \nabla_{\gamma} C_{\beta}, \tag{4.3}$$

$$a_{\gamma\beta} \equiv C_{\gamma}{}^{\alpha}(C_{\alpha\beta} - C_{\beta\alpha}) + C^{\alpha} C_{\lambda} R_{\gamma\alpha,\beta}{}^{\cdots\cdots\lambda} \tag{4.4}$$

$$b_{\gamma} \equiv -R_{\gamma\lambda} C^{\lambda} + \nabla_{\gamma}\{(\mu'_p - 1)\frac{C^{\alpha}C^{\beta}}{C^{\lambda}C_{\lambda}}\} C_{\alpha\beta} \tag{4.5}$$

The last term in b_{γ} is a quadratic form in $C_{\alpha\beta}$ whose coefficients are functions of the C^{α} and S. These functions can be computed by using the identity (we now use $S =$ constant)

$$\nabla_{\gamma}\mu'_p \equiv \mu''_{p^2}\partial_{\gamma} p, \quad \mu''_{p^2}(p, S) \equiv \frac{\partial \mu'(p, S)}{\partial p}.$$

with by the definition of f and the identity 2.4 it holds that

$$\partial_{\gamma} p = (\mu + p) f^{-1}\partial_{\gamma} f = -(\mu + p)(C^{\lambda}C_{\lambda})^{-1} C^{\alpha} C_{\gamma\alpha}.$$

We complete the computation by using 2.1 and 2.8.

The equations 4.1. are not independent, because they satisfy the identities

$$C^{\beta} M_{\gamma\beta} \equiv 0. \tag{4.6}$$

The equations 4.1 and 4.2. are not a well posed system. Instead of the 4×4 equations 4.1 we consider[3] the 4×3 ones:

$$\tilde{M}_{\gamma i} \equiv M_{\gamma i} - \frac{C_i}{C_0} M_{\gamma 0} = 0 \tag{4.7}$$

[3] An analogous procedure is used for the symmetrization of the Euler equations in K.O. Friedrichs 1969 and in Rendall 1992.

The terms in derivatives of $C_{\gamma\lambda}$ in these equations can be written in the following form:

$$C^\alpha \partial_\alpha (C_{\gamma i} - \frac{C_i}{C_0} C_{\gamma 0}) - (\partial_i - \frac{C_i}{C_0}\partial_0)(C^\alpha C_{\gamma\alpha}) \tag{4.8}$$

Lemma 4.1 *The system 4.2,4.7. is equivalent to a FOS (First Order Symmetric) system for $C_{\gamma\alpha}$ with coefficients being functions of the Riemann tensor, the connection and the dynamical velocity C_λ, but not of their derivatives.*

PROOF. The system is quasi diagonal by blocks, each block corresponding to a given value of the index γ. We will write the principal operator of a block by omitting this index. We set

$$U_i \equiv C_{\gamma i} - \frac{C_i}{C_0} C_{\gamma 0}, \quad U_0 \equiv C^\alpha C_{\gamma\alpha} \tag{4.9}$$

and we define the differential operators $\tilde{\partial}_\alpha$ as follows:

$$\tilde{\partial}_0 \equiv C^\alpha \partial_\alpha, \quad \tilde{\partial}_i = \partial_i - \frac{C_i}{C_0}\partial_0 \tag{4.10}$$

The principal terms (derivatives of $C_{\gamma\alpha}$) in the equations 4.7 with index γ are

$$\tilde{\partial}_0 U_i - \tilde{\partial}_i U_0. \tag{4.11}$$

We have by inverting 4.9:

$$C_{\gamma 0} \equiv \frac{C_0(U_0 - C^i U_i)}{C^\lambda C_\lambda}$$

$$C_{\gamma i} \equiv U_i + \frac{C_i(U_0 - C^j U_j)}{C^\lambda C_\lambda}$$

The principal terms of 4.2. read, using the above formulae

$$\frac{\mu'_p C^\alpha \partial_\alpha U_0}{C^\lambda C_\lambda} + (g^{ij} - \frac{C^i C^j}{C^\lambda C_\lambda})\partial_i U_j - \frac{C^0 C^i}{C^\lambda C_\lambda}\partial_0 U_i \tag{4.12}$$

■

We introduce the positive definite (if C is timelike) quadratic form

$$\tilde{g}^{ij} \equiv g^{ij} - \frac{C^i C^j}{C^\lambda C_\lambda}. \tag{4.13}$$

Then we find that

$$\tilde{g}^{ij} \frac{C_j}{C_0} \equiv \frac{C^0 C^i}{C^\lambda C_\lambda}$$

The principal terms 4.12 are therefore

$$\frac{\mu_p' \tilde{\partial}_0 U_0}{C^\lambda C_\lambda} + \tilde{g}^{ij} \tilde{\partial}_i U_j \tag{4.14}$$

The matrix of the coefficients of the derivatives $\tilde{\partial}_\alpha$ in the equations deduced from the system 4.2, 4.7 is

$$\begin{pmatrix} -\frac{\mu_p'}{C^\lambda C_\lambda} \tilde{\partial}_0 & -\tilde{\partial}^1 & -\tilde{\partial}^2 & -\tilde{\partial}^3 \\ -\tilde{\partial}_1 & \tilde{\partial}_0 & 0 & 0 \\ -\tilde{\partial}_2 & 0 & \tilde{\partial}_0 & 0 \\ -\tilde{\partial}_3 & 0 & 0 & \tilde{\partial}_0 \end{pmatrix}$$

it is symmetrized by taking the product with the 4×4 matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \tilde{g}_{ij} \end{pmatrix}.$$

4.1 Hyperbolicity.

A FOS system is hyperbolic with respect to the space slices x^0 =constant if the system is still symmetric when written with the usual partial derivatives, and is such that the corresponding matrix M^t of the coefficients of derivatives $\frac{\partial}{\partial t}$ is positive definite. It admits then an energy inequality for a positive definite energy relative to the space slices. This fact is of physical interest and also for the proof of existence of solutions of the Cauchy problem, at least of solutions local in time,

In the case we are considering the matrix $\tilde{M}^0$ is diagonal, with positive elements if $\mu_p' > 0$ and C^α is timelike. The principal matrix written with natural coordinates is also symmetrizable. The corresponding matrix M^t is not diagonal, and it is not obvious that it is positive definite. In fact it will be so only if $\mu_p' \geq 1$. We will prove the following lemma.

Lemma 4.2 *The system 4.2, 4.7 is FOSH if $\mu'_p \geq 1$ and C is timelike.*

PROOF. It is simpler to compute directly the energy inequality for the considered system: its positivity is equivalent to the positivity of the matrix M^t. Multiplying 4.2. and 4.7 respectively by U_0 and $\tilde{g}^{ij}U_j$ gives equations of the form

$$\frac{1}{2}\frac{\mu'_p\tilde{\partial}_0(U_0)^2}{f^2} - \tilde{g}^{ij}U_0\tilde{\partial}_i U_j = U_0 F_0 \tag{4.15}$$

$$\tilde{g}^{ij}U_j\tilde{\partial}_0 U_i - \tilde{g}^{ij}U_j\tilde{\partial}_i U_0 = \tilde{g}^{ij}U_j F_i$$

where the F_α contain only non differentiated terms. We add these two equations, replace the operators $\tilde{\partial}$ by the operators ∂ and carry out some manipulations using the expression for $\tilde{g}^{ij}$ and the Leibniz rule. We obtain that

$$\partial_0\mathcal{F} + \bar{\nabla}_i\mathcal{H}^i = Q_F$$

The function Q_F is a quadratic form in $C_{\gamma\alpha}$ and the Riemann tensor, while $\mathcal{H}^i$ and $\mathcal{F}$ are given by the following expressions;

$$\mathcal{H}^i = \frac{1}{C^0}\{\frac{1}{2}C^i(\frac{\mu'_p}{f^2}U_0^2 + \tilde{g}^{ij}U_iU_j) - \tilde{g}^{ij}U_0U_j\},$$

$$\mathcal{F} \equiv f^{-2}[(\mu'_p - 1)U_0^2 + (U_0 - C^iU_i)^2] + g^{ij}U_iU_j$$

The energy of the dynamical acceleration ∇C relative to the space slices is the quadratic form $\mathcal{F}$. It is positive definite if $\mu'_p \geq 1$ and C^α is timelike. ∎

Remark 4.3 *The system is hyperbolic in the sense of Leray if $\mu'_p > 0$, but the submanifolds x^0 =constant are 'spacelike' with respect to the fluid wave cone only if the fluid sound speed is less than the speed of light, i.e., $\mu'_p \geq 1$.*

5 Bel-Robinson type energy of the system.

The Bianchi system 3.1,3.2 and analogous equations obtained by replacing $(0j)$ by (ij), together with the system 4.2, 4.7. satisfied by the dynamical

acceleration constitute a FOSH system if $\mu'_p \geq 1$. Its Bel - Robinson type energy (superenergy) density on a space slice is the sum of the Bel - Robinson energy density of the gravitational field, and the energy density of the dynamical acceleration:

$$\mathcal{E} \equiv \mathcal{B} + \mathcal{F}.$$

Using the expression of ∂_0 and the mean extrinsic curvature $\tau \equiv g^{ij} K_{ij}$ of the space slices S_t, whose volume element we denote by $\mu_{\bar{g}_t}$, we obtain an integral equality whose right hand side couples gravitational and fluid superenergies:

$$\int_{S_t} \mathcal{E} \mu_{\bar{g}} = \int_{S_t} \mathcal{E} \mu_{\bar{g}} + \int_{t_0}^{t} \int_{S_\theta} \{-N\tau\mathcal{E} + Q_G + \mathcal{S} + Q_F\} \mu_{\bar{g}} d\theta$$

The scalars $Q_G, \mathcal{S}$ and Q_F can be estimated in terms of $\mathcal{E}$ and thus lead to a linear inequality for $\mathcal{E}$, permitting the estimate of its growth with time.

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YCB. LPTL, Université Paris 6, 4, 75252, Paris. YCB@ccr.jussieu.fr

JWY. Physics Department, Cornell University, Ithaca, NY, 14853-6801, USA. York@astro.cornell.edu